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ON THE COVERING CUTS OF c^d $(d \leq 5)$

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A new proof for the problem of the cut number of the 4-cube is given. This proof is independent of the characterization of the cuts, thus it provides a shorter proof for the problem. This proof and the characterization of the cuts are used to prove that the maximal number of edges of c^4 covered by 3 cuts is 30. Finally, it is shown that up to isomorphism there is only one cut for c^5 that contains edges from all subfacets of the 5-cube.

Preliminaries

We use the terminology of Convex polytopes from Grünbaum [3]. The d-dimensional cube is denoted by c^d. A *cut-complex* C is a sub-complex of $B(c^d)$ for which there exists a hyperplane H that separates strictly vertices of C from rest of the vertices of c^d. An *exterior edge* of C is an edge joining vertices of C and vertices of its graph complement $\bar{C}$. A cut of c^d, is the set of exterior edges of some cut-complex C of c^d. $F_i(x)$ is the set of i-faces of a polytope or a complex x. The *direction* of an exterior edge $e = v_1v_2$ of a complex C is $d(e) = v_2 - v_1$, where $v_1 \in C$ and $v_2 \in \bar{C}$. Let $G_1, G_2, \ldots, G_n$ be n cuts of c^d $(d \geq 3)$. The set of $\{G_1, G_2, \ldots, G_n\}$ is a covering of c^d if they cover all edges of c^d. For a covering $\{G_1, G_2, \ldots, G_n\}$, $\{|G_1|, |G_2|, \ldots, |G_n|\}$ is the set of *edge numbers* of the covering. For example all possible edge numbers of the 3-cube are $\{3,6,3\}$, $\{3,4,5\}$, $\{4,4,4\}$, $\{4,5,5\}$, $\{4,6,6\}$, $\{5,5,5\}$, $\{5,5,6\}$, $\{6,6,5\}$, $\{6,6,6\}$. For a cut G and a vertex v of c^d, $d_G(v)$ is the degree of v in the graph of G, where the vertex v may not be in the graph.

Let F be a face of the d-cube $(\dim F > 1)$, and suppose G and C are a cut and a cut-complex for c^d respectively. Then their restriction to F are a cut and a cut-complex for F and are denoted by G_F and C_F.

The cut number $k(c^d)$ of c^d is the minimal number of cuts needed to cover all edges of the d-cube. It is known that there are 14 different cuts up to isomorphism for the 4-cube by applying this characterization the problem of cut-numbers is solved for $d = 4$, i.e., $k(c^4) = 4$ see [2].

Here we given an alternative proof for the problem above in which the list of different cuts are not used. This proof provides a stronger theorem, which gives

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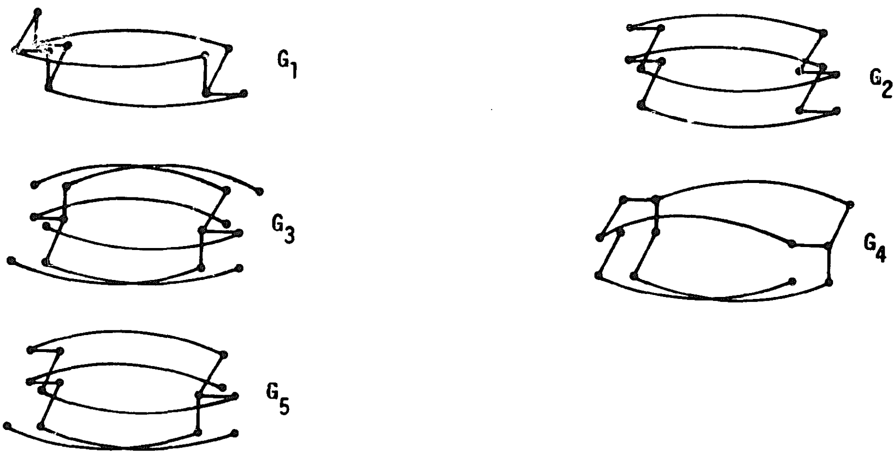

Fig. 1.

the maximal number of edges of c^4 covered by 3 cuts. Note that the cut-number theorem implies this maximal number is less than 32. For the second theorem the set of all cuts of c^4 with 12 edges, i.e., $G_1, G_2, \ldots, G_5$ are used as shown in Fig. 1.

The following lemmas are helpful to prove the main result.

Lemma 1. *For a cut complex C of c^d and parallel exterior edges e_1, e_2 we have $d(e_1) = d(e_2)$.*

Proof. See [2]. □

Lemma 2. *For a k-face F $(0 < k < d)$ of c^d, a cut-complex C and its corresponding cut G, let $v \in F_0(\bar{C}_F)$. Suppose $v_1, v_2, \ldots, v_k$ are all neighbors of v in F. If C_F has exterior edges with directions $v - v_1, v - v_2, \ldots, v - v_k$, then $d_{G_F}(v) < d_G(v)$ implies that G is missing all edges of $\bar{F}$, where $\bar{F}$ is some k-face parallel to F.*

Proof. $d_{G_F}(v) < d_G(v)$ implies that there is some exterior edge of F incident to v (say $v\bar{v}$) in G, then $\bar{v} \in C$. Let $\bar{F}$ be the k-face parallel to F containing $\bar{v}$. Now by Lemma 1 all neighbors of $\bar{v}$ in $\bar{F}$ lie in C. By repeating this argument, it can be seen that all vertices of $\bar{F}$ also lie in C, therefore $\bar{F} \in C$. $\bar{F}$ is the desired face of c^d. □

In the following lemma F is a 3-face and G is a cut of c^d $(d \geqslant 4)$, which contains edges from all 3-faces of c^d.

Lemma 3. (a) *If G contains 3 edges of F that are incident to a vertex w, then $d_{G_F}(v) = d_G(v) = 0$, where v is the opposite vertex of w in F.*

(b) *If G contains 4-edges of F that have vertices v_1 and v_2 of valence 2 in G_F then $d_G(v_1) = d_G(v_2) = 2$.*

(c) *If G contains* 5*-edges of F, then for some vertex v of F*, $d_G(v) = d_{G_F}(v) = 1$.

(d) *If G contains* 6*-edges of F, then F contains two opposite vertices* v_1, v_2 *such that* $d_G(v_1) = d_G(v_2) = 0$.

(e) *G cannot contain* 4*-parallel edges of F.*

Proof. (a) Choose the cut-complex C of G with $\bar{C}_F = F - \{w\}$. Now if $d_{G_F}(v) < d_G(v)$, then, by Lemma 2, G misses a 3-face $\bar{F} \parallel F$ which contradicts the property of G above.

(b) If $d_G(v_1) > 2$, then there is some exterior edge of F say $v_1\bar{v}_1$ in G. If we apply Lemma 1 to the 4-cube containing F and $v_1\bar{v}_1$ and the cut-complex of G missing v_1, v_2, the desired contradicting 3-face, will be obtained. The same argument works for v_2.

(c) Obviously one of the cut-complexes of G_F has only two edges, say $\bar{C}_F = \{vv_1, vv_2\}$. Now apply Lemma 2 for C, which implies $d_G(v) = d_{G_F}(v) = 1$.

(d) The same argument works for G_F and both vertices v_1, v_2 of F with $d_{G_F}(v_1) = d_{G_F}(v_2) = 0$.

(e) Let v_1v_1', v_2v_2', v_3v_3', v_4v_4' be 4 parallel edge of F in G. We claim that G does not contain any exterior edge of F. Suppose $v_1\bar{v}_1 \in G$, and consider the 4-cube $\mathrm{Conv}\{F, \bar{F}\}$, where $\bar{F}$ is the 3-face parallel to F containing $\bar{v}_1$. If $\bar{v}_i$, $\bar{v}_i'$ are vertices of $\bar{F}$ corresponding to v_i, v_i', $i = 1, 2, 3, 4$ respectively, then $\bar{v}_1$, v_1', v_2', v_3', v_4' lie in the same cut-complex C of G. So by Lemma 1 $\bar{v}_i' \in C$ for $i = 1, 2, 3, 4$, therefore G is missing the 3-face $\mathrm{Conv}\{v_i', \bar{v}_i' : 1 \leq i \leq 4\}$, which is a contradiction. But since G contains the 4 parallel edges of F and does not contain any exterior edges of F, it has to miss two parallel 3-faces of the 4-cube, which again contradicts the property of G. □

In the following a vertex is isolated for a cut G if $d_G(v) = 0$.

Lemma 4. *Let F be a* 3*-face of the d-cube* c^d $(d \geq 4)$, *and suppose* $\{G, G', G''\}$ *is a set of cuts of* c^d *which covers all edges and all exterior edges of F, moreover the* 3 *cuts contain edges from all* 3*-faces of* c^d. *Then neither the edge number set of* $\{3, 6, 3\}$ *nor* $\{3, 4, 5\}$ *can be produced on the face F, by the* 3 *cuts.*

Proof. Suppose $\{3, 6, 3\}$ is the edge number set of $\{G, G', G''\}$ restricted to F, say, G_F, G'_F, G''_F have 3, 6 and 3 edges respectively. If v_1, v_2 are the two isolated vertices of F for G' these are isolated also for one of the G or G''. So all edges incident to v_1 or v_2 must be covered by the third cut which is impossible. The last statement follows from Lemma 3 and the structure of $\{3, 6, 3\}$ on F.

Now suppose $\{3, 4, 5\}$ is produced by G, G', G'' respectively. Again by the same argument there is some vertex v of F such that $d_G(v) = 0$, $d_{G'}(v) = d_{G'_F}(v) = 2$ and $d_{G''}(v) = d_{G''_F}(v) = 1$ so exterior edges of F incident to v are not covered by any of the 3 cuts, which is a contradiction. □

Now we are ready for

Theorem 1. $k(c^4) = 4$.

Proof. Suppose there are 3 cuts G, G', G'' covering all edges of the 4-cube. It can be shown that each of the 3 cuts have to contain edges from all facets of the 4-cube. But by Lemmas 3 and 4, the edge number set $\{3,6,3\}$, $\{3,4,5\}$ and $\{4,4,4\}$ cannot be produced on any 3-face, and the rest of edge number sets of a 3-cube each as at least 2 overlaps so there are 4 pairs of parallel facets each with at least 4 overlaps, i.e., there exists more than 4 overlaps which is impossible. (The 3 cuts have 36 points of intersections with hyperplanes and $|F_1(c^4)| = 32$.) □

Now we use the proof above and the characterization of the cuts for c^4 to give a generalization of Theorem 1.

Theorem 2. *The maximum number of edges of c^4 that can be covered by* 3 *cuts is* 30.

Proof. First we show that there are no 3 cuts covering 31 edges of c^4, this can be seen among lines of the proof of Theorem 1. Again if there are 3 cuts covering 31 edges of c^4, they have to contain edges from all facets of the 4-cube. The edge number sets of $\{4,4,4\}$ and $\{3,6,3\}$ are rejected by the same argument as of Theorem 1. Now suppose $\{3,4,5\}$ is produced by G, G', G'' respectively, let v be the same vertex as before, i.e., $d_G(v) = 0$, $d_{G'}(v) = d_{G'_F}(v) = 2$ and $d_{G''}(v) = d_{G''_F}(v) = 1$. If w is the opposite vertex to v in F, obviously $d_G(w) = d_{G_F}(w) = 3$ and $d_{G''}(w) = d_{G''_F}(w) = 0$. Since the exterior edge of F incident to v cannot be covered by the 3 cuts therefore $d_{G'}(w) = 1$ so $G' = G_5$.

Since G_5 contains just 3 edges of the 3-face $\bar{F}$ parallel to F, then the edge number set $\{3,4,5\}$ is the only one acceptable for $\bar{F}$. Therefore there is some vertex $\bar{v} \in \bar{F}$ with the property that, the exterior edge of $\bar{F}$ incident to it cannot be covered by the 3 cuts. The structure of G_5 shows that v and $\bar{v}$ are not adjacent, so there are at least 2 edges that are missed by the cuts, which is impossible. We have to show that there are 3 cuts covering 30 edges of the 4-cube, to do this, choose the 3-cuts G_3, G_2, G_3.

It is easy to see that the 3 cuts can be embedded in the 4-cube such that they cover all edges of c^4 except two parallel edges connecting F and $\bar{F}$. In fact they will produce the edge number set of $\{3,6,3\}$ for both of F and $\bar{F}$. □

Remark. If we wanted to use the characterization of the cuts of c^4, we would not need Lemmas 1, 2 and 3. Therefore, we give here, a shorter proof for Theorem 1 than the original one, see [2]. The problem of the cut number for c^5 involves those cuts that contain edges from all facets of the 5-cube.

The following theorem determines a sub-class of these cuts, i.e., those that contain edges from all sub-facets of c^5. Consider H_0, the cut of the 5-cube corresponding to the following hyperplane in R^5: $\{(x_1, \ldots, x_5): \sum_{i=1}^{5} x_i = \frac{5}{2}\}$, where $c^5 = \{(\varepsilon_1, \ldots, \varepsilon_5): \varepsilon_i = 0, 1\}$. Obviously H_0 contains edges from all 3-faces of the 5-cube. Now we have:

Theorem 3. *H_0 is the only cut of c^5, up to isomorphism, that does not miss any sub-facet of the cube.*

Proof. Let H be a cut of c^5 with the property above and with corresponding cut-complexes C and $\bar{C}$. Moreover, let $F = \mathrm{Conv}\{F_1, F_2\}$ be a facet of c^5 where F_1 and F_2 are two parallel facets of F. We claim that $H_F \neq G_2$, where G_2 is given in the list of the cuts of c^4. Suppose $|H_{F_1}| = |H_{F_2}| = 6$, $v_1, v_2 \in C$ and $v'_1, v'_2 \in \bar{C}$, where v_i and v'_i are isolated vertices of H in F_i, $i = 1, 2$ respectively. These vertices are also isolated in the two parallel 4-cubes T, U containing F_1 and F_2 respectively. Let $\bar{F}_1$ and $\bar{F}_2$ be the 3-faces of T and U parallel to F_1 and F_2 respectively. Therefore $\bar{v}_1, \bar{v}_2 \in C$, where $\bar{v}_i$ is the neighbor of v_i in $\bar{F}_i$ $i = 1, 2$. Now choose two parallel edges $\bar{v}_1 w_1$, $\bar{v}_2 w_2$ from $\bar{F}_1$ and $\bar{F}_2$ respectively. If $w_1, w_2 \in C$, then the 3-face containing v_1, v_2, $\bar{v}_1$, $\bar{v}_2$ and w_1 stays in C which is impossible. So one of w_1 or w_2, say w_1, stays in $\bar{C}$. Lemma 1 implies the 3-face of the 5-cube containing v'_1 and 4 Parallel edges from F_1, F_2, $\bar{F}_1$, $\bar{F}_2$ respectively stays in $\bar{C}$. Therefore either case of $w_1 \in C$ or $\bar{C}$ leads to a missing 3-face, which is a contradiction and proves our claim. In the list of the cuts of c^4 with 12 edges, G_2 is the only one with $|(G_2)_F| \neq 3$ for every 3-face F. Then $|H_{F_1}| = 3$ for some 3-face F_1 of c^5. Let F_2, F, $\bar{F}_1$, $\bar{F}_2$, T, U be defined as before, and suppose C is the cut complex of H with $C_{F_1} = \{a\}$, where a is the common vertex of the edges in H_{F_1}. Now consider the neighbors of a say b, c, d and its opposite vertex h in F_1. h is isolated for H and H contains all exterior edges of F_1 incident to b, c, d and is missing all exterior edges of F_1 incident to a.

Let a_1, b_1, c_1, d_1, h_1 be neighbors of a, b, c, d, and h in F_2 respectively, and define $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d}$, $\bar{h}$ in $\bar{F}$ by the same manner. Therefore a_1, b_1, c_1, d_1, $\bar{a}$, $\bar{b}$, $\bar{c}$, $\bar{d} \in C$, and $h_1, \bar{h} \in \bar{C}$.

Now consider facets $\mathrm{Conv}\{\bar{F}_1, \bar{F}_2\}$ and $\mathrm{Conv}\{F_2, \bar{F}_2\}$. Then Lemma 1 and the fact that H does not miss any 3-face of the 5-cube imply that all neighbors of h_1 in $F_2 \in \bar{C}$. By symmetry also all neighbors of $\bar{h}$ in $\bar{F}_1 \in \bar{C}$, so $|H_{F_2}| = |H_{\bar{F}_1}| = 6$.

Finally the same argument shows the neighbor of h_1 in $\bar{F}_2$ is the only vertex of $\bar{F}_2$ in $\bar{C}$. Then $|H_{\bar{F}_2}| = 3$ and H is isomorphic to H_0. □

Corollary. *If $k(c^5) = 4$, then the 4 cuts have to contain edges from all facets of c^5, but at least one of the cuts is missing one or more subfacets of the 5-cube.*

Proof. The first part is a conclusion of Theorem 1 and the second is an application of the last theorem. In fact it is easy to check that there are no 4 copies of H_0 that can cover all edges of a 3-face and all of its exterior edges in c^5.

Final remarks

It is important to note that there are more open problems related to the content of this paper. It seems that nothing about cuts or cut numbers is known in dimensions greater than 4.

In [2] cut-complexes of the 4-cube are characterized by an algorithm which can be improved for dimension $d \leq 8$; It is well-known that for two sets $M_1, M_2 \subset F_0(c^d)$ $(d \leq 8)$, if every two vertices of M_1 are separable from every two vertices of M_2, then M_1 and M_2 are separable by a hyperplane. This fact which releases the algorithm of [2] from many calculations was not known to the author at the time the paper went to print.

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